Physics B I

Answer the following problems on quantum mechanics.

I − a　Answer the following problems regarding the orbital angular momentum operators.

(1) The x component of angular momentum l_x is written as $l_x = yp_z - zp_y$. In quantum mechanics, momentum operator $\hat{p}_x$ is represented as

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

where $\hbar = h/2\pi$ is the Planck constant. Calculate the commutation relation between orbital angular momentum operator $\hat{l}_x$ and position operators $(\hat{x}, \hat{y}, \hat{z})$, that is $[\hat{l}_x, \hat{x}]$, $[\hat{l}_x, \hat{y}]$, and $[\hat{l}_x, \hat{z}]$. The commutation of operators $\hat{A}$ and $\hat{B}$ is given by $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

(2) Derive the equation of $[\hat{l}_x, \hat{l}_y] = i\hbar\hat{l}_z$.

(3) For arbitrary operators $\hat{A}$ and $\hat{B}$, the equation

$$[\hat{A}^2, \hat{B}] = \hat{A}[\hat{A}, \hat{B}] + [\hat{A}, \hat{B}]\hat{A}$$

holds. Prove the square of orbital angular momentum operator $\hat{l}^2$ and the z component of orbital angular momentum operator $\hat{l}_z$ commute to each other.

I − b　We consider the system of a quantum particle with mass m locating in a box potential of $V = 0$ in the region of $0 \le x \le L$ and $V = \infty$ in the region of $x < 0, L < x$. The wavefunction and its eigenvalue of this system are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right),\ E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}\ (n=1, 2, \ldots) \quad (1)$$

in the region of $0 \le x \le L$.

Answer the following problems.

(1) Write down the Schrödinger equation of this system.

(2) Write down the boundary conditions for the wavefunction.

(3) Derive the solutions of equation (1) by assuming the wavefunction of this system as

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

and applying the boundary conditions for the wavefunction.

Physics B II

Answer the following problems on statistical mechanics.

Ising spin in magnetic field has two energy values $-\varepsilon$ or ε.

(1) When the energy of state i is given by E_i, partition function Z in the canonical ensemble at temperature T is given by

$$Z = \sum_{\text{all states } i} \mathrm{e}^{-\frac{E_i}{kT}},$$

where k is the Boltzmann constant. Calculate partition function Z of one Ising spin.

(2) In the following problems, we consider a system of N independent Ising spins. Calculate partition function Z of this system in the canonical ensemble at temperature T.

(3) Calculate free energy F using $F = -kT \log Z$.

(4) Calculate entropy S using $S = -\frac{\partial F}{\partial T}$.

(5) Calculate internal energy U using $U = F + ST$.

(6) Calculate heat capacity C using $C = \frac{\partial U}{\partial T}$ and plot C as a function of T.