Physics B I

Solve the following problems on quantum mechanics.

The relation $p = \hbar k$ ($\hbar = \dfrac{h}{2\pi}$ where h is the Planck constant) holds between the de Broglie wavenumber k and momentum p of a particle of mass m. Consider the situation where this particle enters the potential $V(x)$ described in the figure on the next page from the left along the x axis with its initial energy E ($>V_0>0$). The wave functions of this particle in Regions 1 and 2 are descibed with an incoming plane wave $\psi_{\text{in}}(x) = Ae^{ik_1x}$, a reflected plane wave $\psi_{\text{out}}(x) = Be^{-ik_1x}$, and a transmitted plane wave $\psi_{\text{trans}}(x) = Ce^{ik_2x}$ as follows.

$$\Psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$$
$$\Psi_2(x) = Ce^{ik_2x}$$

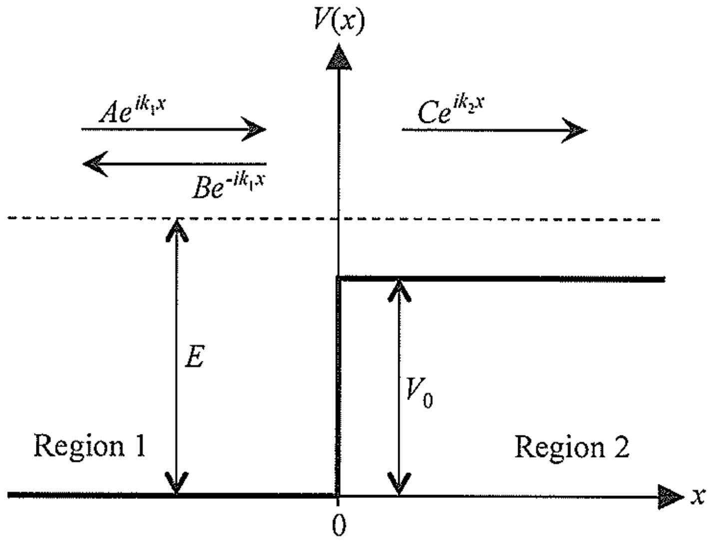
(1) Find the values of k_1 and k_2 in terms of m, E, V_0, and $\hbar$.

You may use k_1 and k_2 if necessary in the answers of the following problems.

(2) Find the relations among A, B, and C so that $\Psi_1(x)$ and $\Psi_2(x)$ will be connected smoothly at $x = 0$.

(3) The probability current j of a plane wave ψ is given by $j = \dfrac{\hbar}{2im}\left(\psi^* \dfrac{d}{dx}\psi - \psi \dfrac{d}{dx}\psi^*\right)$ (ψ^* is the complex conjugate of ψ). Find the values of a reflection probability R and a transmission prbability T at $x = 0$, and explain how they are different from what are expected from classical mechanics.

(4) Describe the condition under which the reflection probability R converges to zero.



$V(x)$
Ae^{ik_1x}
Be^{-ik_1x}
Ce^{ik_2x}
E
V_0
Region 1
Region 2
0
x

Physics B II

Solve the following problems on statistical mechanics.

Consider a system that N one-dimensional harmonic oscillators (mass m, angular frequency ω) are in thermal equilibrium at temperature T. The total energy E of this system is

$$E = \sum_{i=1}^{N} \left(\frac{1}{2m} p_i^2 + \frac{m}{2} \omega^2 x_i^2 \right),$$

where x_i and p_i are the position and momentum of a particle i.
In this case, the partition function Z in the canonical ensemble is given by

$$Z = \frac{1}{h^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-E/kT} \mathrm{d}^N p \, \mathrm{d}^N x.$$

Here, h is the Planck constant, k is the Boltzmann constant, $\mathrm{d}^N p = \mathrm{d}p_1 \cdots \mathrm{d}p_N$, and

$\mathrm{d}^N x = \mathrm{d}x_1 \cdots \mathrm{d}x_N$.

(1) Calculate the partition function Z. You may use the following relation, if necessary.

$$\int_{-\infty}^{\infty} e^{-ax^2} \mathrm{d}x = \sqrt{\frac{\pi}{a}} \quad (a > 0)$$

(2) Calculate the Helmholtz free energy F using the following relation.

$$F = -kT \log Z$$

(3) When the internal energy and the entropy are expressed by U and S, respectively, the following relations hold.

$$U = F + TS, \quad S = -\left(\frac{\partial F}{\partial T} \right)_V$$

Derive the following relation.

$$U = -T^2 \left[\frac{\partial}{\partial T} \left(\frac{F}{T} \right) \right]_V$$

(4) Calculate the internal energy U and the heat capacity at constant volume $C_V = \left(\frac{\partial U}{\partial T} \right)_V$ of this system.