Physics A I

Answer the following questions on classical mechanics.

I

Answer problems of motion of the earth moving around the sun and derive Kepler's laws of planetary motion. Assume that a system consists of only the earth and the sun (no influence by external forces). Also, there is no frictional resistance. Let m_1, m_2, $\mathbf{r}_1$ and $\mathbf{r}_2$ be the masses and position vectors of point mass 1 (the earth) and point mass 2 (the sun), respectively. Relative position vector of the point mass 1 with regard to the point mass 2 is defined by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Internal forces acting at the point masses 1 and 2 are $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, respectively, and there is a relation between them: $\mathbf{F}_{12} = -\mathbf{F}_{21} = \mathbf{F}(\mathbf{r})$.

(1) Position vector of center of mass for the system is defined as follows:

$$\mathbf{r}_C = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}.$$

Write down equation of motion for the center of mass.

(2) Derive equation of motion for the relative motion of two-body system with the relative position vector $\mathbf{r}$ and the reduced mass μ, defined by $\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}$.

(3) Since the internal force $\mathbf{F}(\mathbf{r})$ is a central force, it is described as $\mathbf{F}(r) = f(r)\frac{\mathbf{r}}{r}$, where $f(r)$ is an arbitrary function. By considering this point, explain the reason why angular momentum of the relative motion, $\mathbf{L} = \mathbf{r} \times \mu \frac{d\mathbf{r}}{dt}$, is conserved.

(4) When the z-axis is directed to the angular momentum, $\mathbf{L}$, the relative motion is in the two-dimensional (2D) plane perpendicular to the z-axis. By introducing 2D polar coordinates (r, θ), Cartesian coordinates can be converted with

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}.$$

In the polar coordinates, velocity and acceleration components can be described as follows:

$$\begin{cases} v_r = \dfrac{dr}{dt} \\ v_\theta = r\dfrac{d\theta}{dt} \end{cases}, \quad \begin{cases} a_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 \\ a_\theta = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) \end{cases}.$$

Equation of motion for the relative motion in the 2D polar coordinate system is similarly described as follows:

$$\begin{cases} \mu a_r = \mu\left[\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2\right] = F_r \\ \mu a_\theta = \mu\left[\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)\right] = F_\theta \end{cases}.$$

In the two-body system consisting only of the earth and the sun, r and θ components of the internal force $\mathbf{F}(\mathbf{r})$ are described as $F_r = f(r)$ and $F_\theta = 0$, because it is a central force. The equation of motion shows that magnitude of angular momentum, $L \equiv \mu r^2 \dfrac{d\theta}{dt}$, is constant.

(4a) Universal gravitation of $f(r) = G\dfrac{m_1 m_2}{r^2}$ acts in the relative motion (G: gravitational constant). Derive mechanical energy E in a form including the L. Note that a kinetic energy part in the mechanical energy E is described with $\dfrac{1}{2}\mu\left\{\left(\dfrac{dr}{dt}\right)^2 + r^2\left(\dfrac{d\theta}{dt}\right)^2\right\}$. The second term in the kinetic energy part corresponds to contribution from centrifugal force.

(4b) Illustrate effective potential energy V_{eff} including the centrifugal force term as a function of relative distance r, schematically. Also, derive a minimal value V_{m} of the potential energy V_{eff} and a distance d which gives the minimal value, and indicate the derived values of the V_{m} and the d in the illustration of the potential energy V_{eff}.

(4c) The equation $r = \dfrac{d}{1+\varepsilon\cos\theta}$ is obtained as an orbital equation (relation between r and θ), where eccentricity is given by $\varepsilon = \sqrt{1 - E/V_{\mathrm{m}}}$. Convert the orbital equation with the Cartesian coordinates (x, y), and find a condition on the eccentricity ε that the earth's orbit is an ellipse. Also, derive major and minor radii of the elliptical orbit, a and b, in terms of the eccentricity ε and the d obtained in question (4b) (Kepler's first law).

(5) Kepler's second law states that the radius vector from the sun sweeps out equal areas in equal periods of time (dS/dt = constant, S: sweeping area of the earth's orbit around the one of the two foci). Prove this law. Infinitesimal change of the swept area can be approximated with $dS \cong \dfrac{1}{2}(r d\theta)\cdot r$ against tiny variation of angle, $d\theta$.

(6) Derive Kepler's third law which states that the square of the period of orbital motion of the earth is simply proportional to the cube of the major radius of the earth's orbit.

Physics A II

Answer the following questions concerning electromagnetism.

II

You should use MKSA units in all the questions and the permittivity of vacuum given by ε_0.

(1) As shown in Fig. 1, electric charges of q and $-q$ are located along the z-axis at $z = a/2$ and $z = -a/2$, respectively with a distance of a. Calculate electrostatic potential $\phi(\vec{r})$ at $\vec{r} = (x_0, y_0, z_0)$ generated from these charges in terms of r_+ and r_- where r_+ and r_- are distances from the charges of q and $-q$, respectively and the position of $\vec{r}$ is far from the charges. Here, the electrostatic potential is set to be zero at an infinite distance.

(2) An electric dipole 1 with a moment of $\vec{p}_1 = q\vec{a}$ is defined for the pair of the two charges as shown in Fig.2. Show that the electrostatic potential obtained in the question (1) can be rewritten in the form of $\phi(\vec{r}) = \dfrac{\vec{p}_1 \cdot \vec{r}}{4\pi\varepsilon_0 r^3}$. To derive this equation, you should use the relation of $r^2 >> a^2$, assuming that the intercharge distance a is sufficiently short and the position of $\vec{r}$ is far from the electric dipole. θ is the angle between the z-axis and the vector $\vec{r}$ and r is the length of $\vec{r}$.

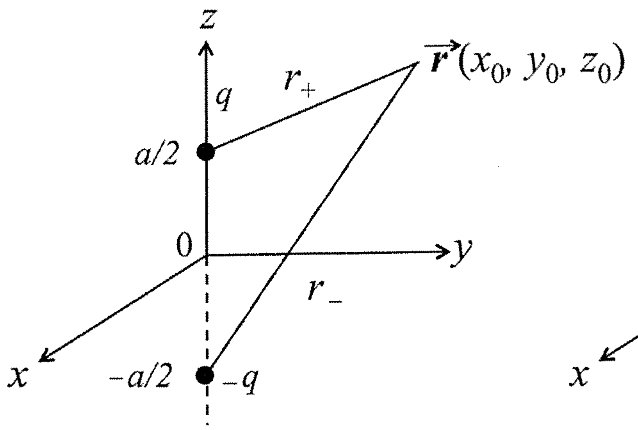

Fig. 1

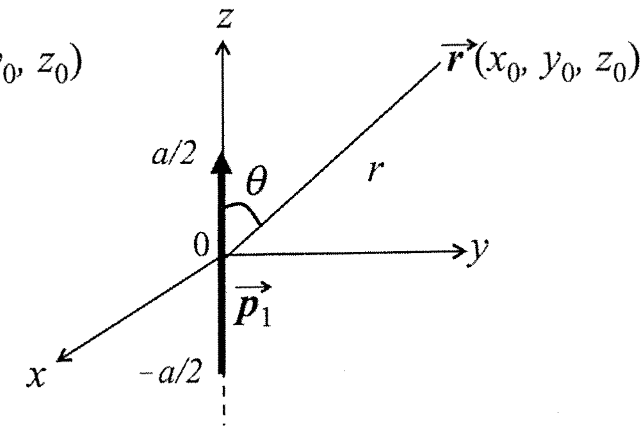

Fig. 2

(3) By using the electrostatic potential given in the question (2), calculate the electric field $\vec{E}_1(\vec{r})$ at $\vec{r}$.

(4) Suppose that another electric dipole 2 with a moment of $\vec{p}_2 = q\vec{b}$ is located at $\vec{r}$ and the

electric field $\vec{E}_1(\vec{r})$ calculated in the question (3) exerts on this dipole 2. We here consider the potential energy U of the electric dipole 2. The potential energy can be written in the form of $U = q\phi(\vec{r}+\frac{\vec{b}}{2}) - q\phi(\vec{r}-\frac{\vec{b}}{2})$ where the center of the dipole 2 is set to $\vec{r}$ and charges of q and $-q$ are located at $\vec{r}+\frac{\vec{b}}{2}$ and $\vec{r}-\frac{\vec{b}}{2}$, respectively. Write down U in terms of $\vec{p}_2$ and $\vec{E}_1(\vec{r})$ by performing the Taylor expansion and considering up to the first-order of $\vec{b}$, assuming that the interchange distance b of the electric dipole 2 is sufficiently short compared with $|\vec{r}|$.

(5) Show that U obtained in the question (4) can be rewritten in the form of $U = \frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}_1\cdot\vec{p}_2}{r^3} - \frac{3(\vec{p}_1\cdot\vec{r})(\vec{p}_2\cdot\vec{r})}{r^5}\right]$. Note that this energy U is called a dipole-dipole interaction.