Physics B I

Answer the following questions concerning quantum mechanics.

Consider a one-dimensional particle which is confined between two rigid walls. The potential $V(x)$ is written as follows,

$$V(x) = \begin{cases} 0 & (0 \leq x \leq L) \\ \infty & (x < 0, x > L). \end{cases} \tag{1}$$

Here the mass of the particle is m, the Planck constant is h, and $\hbar = h/2\pi$.

(1) Write down the Schödinger equation for the particle in the region $0 \leq x \leq L$, assuming that the energy of the particle is E.

(2) Find the eigen functions $\phi_0(x)$ and $\phi_1(x)$ for the ground and first excited states of the particle as well as the eigen energies E_0 and E_1, respectively. Each eigen function should be real and normalized.

(3) Evalulate the expectation values of x for the ground and first excited states, respectively.

(4) In general, the time-dependent wave function $\psi_n(x,t)$ of an eigen function $\phi_n(x)$ can be written as $\phi_n(x)\exp(-iE_nt/\hbar)$. Suppose that the wave function of the particle is $\Psi(x,t)$ and the initial state $\Psi(x,0)$ is given by $\frac{1}{\sqrt{2}}\phi_0(x) + \frac{1}{\sqrt{2}}\phi_1(x)$. Evaluate the time-dependent expectation value of x and the period of the oscillation.

Physics B II

Solve the following problems regarding statistical thermodynamics.

Let a particle with spin $\frac{1}{2}$ be fixed in the magnetic field H $(H > 0)$. The spin can be either paralleled or anti-paralleled to the magnetic field. When the spin is paralleled to the field, the energy is given as $E_- = 0$. On the other hand, when the spin is anti-paralleled to the field, the energy is given as $E_+ = 2\mu_0 H$, where $\mu_0 > 0$ denotes the spin magnetic moment.

In what follows, we consider that a system composed of N particles with spin $\frac{1}{2}$ and the magnetic moment $\mu_0 > 0$ are located in the uniform magnetic field H $(H > 0)$. The particles do not interact with each other. When the system is in equilibrium at temperature T, the partition function of the canonical ensemble, Z, is expressed as follows:

$$Z = \prod_{n=1}^{N} \sum_{\sigma=+,-} \exp\left(-\frac{E_\sigma^{(n)}}{k_\mathrm{B} T}\right),$$

where k_B denotes the Boltzmann constant.

(1) Express the partition function Z in terms of N, $\varepsilon = 2\mu_0 H$, and $\beta = (k_\mathrm{B} T)^{-1}$.

(2) Express the internal energy U in terms of N, $\varepsilon = 2\mu_0 H$, and $\beta = (k_\mathrm{B} T)^{-1}$.

(3) Show that the heat capacity C_V is expressed as $C_\mathrm{V} = -\dfrac{1}{k_\mathrm{B} T^2}\left(\dfrac{\partial U}{\partial \beta}\right)_{V,N}$.

(4) Express the heat capacity C_V in terms of N, $\varepsilon = 2\mu_0 H$, k_B and T.

(5) Explain the qualitative behaviors of the heat capacity C_V to temperature changes in the low-temperature limit and in the high-temperature limit, together with their physical meanings.

(6) Sketch the graph of the heat capacity C_V as a function of temperature T.