Physics B I

Answer the following questions regarding quantum mechanics.

Consider a particle with mass m and total energy E incident to the one-dimensional potential $V(x)$ shown in Figure 1 from the $x < 0$ region toward the $x > 0$ direction. $V(x)$ is described as follows:

$$V(x) = \begin{cases} 0 & (x < 0) \\ V_0 & (0 \leq x \leq a) \\ 0 & (a < x) \end{cases}$$

In what follows, h denotes the Planck constant and $\hbar = \frac{h}{2\pi}$.

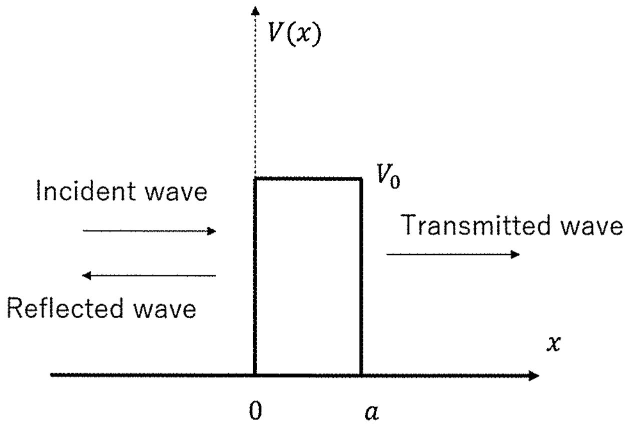

Figure 1 One-dimensional potential

(1) Write the time-independent Schrödinger equations of the particle in the respective regions: $x < 0$ (region I), $0 \leq x \leq a$ (region II), and $a < x$ (region III). Use $\phi(x)$ as the symbol of the wavefunction.

(2) In the case of $V_0 > E > 0$, the wavefunctions $\phi_{\mathrm{I}}(x)$, $\phi_{\mathrm{II}}(x)$, and $\phi_{\mathrm{III}}(x)$ as the solutions of the time-independent Schrödinger equations of (1) for region I, II, and III, respectively, can be expressed as follows:

$$\phi_{\mathrm{I}}(x) = Ae^{ik_1x} + Be^{-ik_1x},$$

$$\phi_{\mathrm{II}}(x) = Ce^{k_2x} + De^{-k_2x},$$

$$\phi_{\mathrm{III}}(x) = Fe^{ik_1x}\ .$$

Give the expressions of positive real numbers k_1 and k_2. You can use m, E, V_0, and $\hbar$ if necessary.

(3) The wavefunctions should connect smoothly at $x = 0$ and $x = a$. Considering this condition, give the expressions of A, B, C, and D that appear in (2) using F, k_1, k_2, and a.

(4) Explain the difference between the behaviors of a classical particle and a quantum particle incident to the potential shown in Figure 1.

Physics B II

Solve the following problems on statistical mechanics.

Consider a system at rest under no external force. We will find the relation between the heat capacity at constant volume and the fluctuation of the total energy of this system. The partition function Z in the canonical ensemble at temperature T is given by

$$Z = \int dE \, \mathrm{e}^{-\beta E} \, \Omega(E),$$

where E is the total energy of this system, $\Omega(E)$ is the density of states, $\beta = \frac{1}{kT}$, and k is the Boltzmann constant. Ensemble average of a physical quantity $A(E)$ is given by

$$\langle A \rangle = \frac{\int dE \, A(E) \, \mathrm{e}^{-\beta E} \, \Omega(E)}{\int dE \, \mathrm{e}^{-\beta E} \, \Omega(E)}.$$

(1) Show that the ensemble average of the total energy $\langle E \rangle$ is given by

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z.$$

Here, log is the natural logarithm.

(2) Derive the following equation:

$$\langle E^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}.$$

(3) Find the relation between $\frac{\partial \langle E \rangle}{\partial \beta}$ and the fluctuation of the total energy $\left\langle \left(E - \langle E \rangle\right)^2 \right\rangle$.

(4) The heat capacity at constant volume C_V is given by $C_V = \frac{\partial \langle E \rangle}{\partial T}$. Find the relation between $\frac{\partial \langle E \rangle}{\partial \beta}$ and C_V.

(5) Find the relation between C_V and $\left\langle \left(E - \langle E \rangle\right)^2 \right\rangle$.