Physics B I

【2 pages in total】

I.

Answer the following questions on quantum mechanics. Let i be the imaginary unit and $\hbar$ be the Planck constant h divided by 2π.

Consider a quantum system described by a time-independent Hamiltonian, $\hat{H}$.

(1) When the state vector at time $t = t_0$ is $|\psi_0\rangle$, the state vector at time t $(t > t_0)$ is expressed as $|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar}|\psi_0\rangle$. The expectation value of a time-independent physical quantity F at time t is obtained using the corresponding operator $\hat{F}$ as

$$\langle F\rangle_t = \langle\psi(t)|\hat{F}|\psi(t)\rangle.$$

When $|\psi_0\rangle$ is the eigenstate belonging to an eigenvalue of the Hamiltonian, find the expectation value, $\langle F\rangle_t$. Let the eigenstate and eigevalue be $|\varphi_n\rangle$ and ε_n, respectively.

(2) Briefly explain the physical reason for the time dependence of $\langle F\rangle_t$ obtained in Question (1).

(3) When we consider the time evolution of a quantum system, we can also consider the operator $\hat{F}$ to be time-evolving, such that $\hat{F}(t) = e^{i\hat{H}(t-t_0)/\hbar}\hat{F}e^{-i\hat{H}(t-t_0)/\hbar}$. When $\hat{F}$ is time-independent, show that the following equation holds:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{F}(t) &= \frac{i}{\hbar}[\hat{H}, \hat{F}(t)] \\ &= \frac{i}{\hbar}e^{i\hat{H}(t-t_0)/\hbar}[\hat{H}, \hat{F}]e^{-i\hat{H}(t-t_0)/\hbar},\end{aligned}$$

where $[\hat{A}, \hat{B}]$ stands for the commutator of operators, $\hat{A}$ and $\hat{B}$.

Next, we consider a particle of mass m in one-dimensional space. Let x and p be its coordinates and momentum, respectively.

(4) Find the commutation relation $[f(\hat{x}), \hat{p}]$ satisfied by the arbitrary function of the coordinate operator $f(\hat{x})$ and the momentum operator $\hat{p}$. When using the coordinate representation, the operators for the coordinate and the momentum are expressed as $\hat{x} = x$ and $\hat{p} = -i\hbar\,\mathrm{d}/\mathrm{d}x$, respetively.

(5) Let the Hamiltonian of this particle be $\hat{H} = \dfrac{1}{2m}\hat{p}^2 + V(\hat{x})$ with the potential function, $V(x)$. Find $\dfrac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}\hat{p}(t)}{\mathrm{d}t}$.

(6) Briefly discuss the results of Question (5) and their relation to classical mechanics.

(7) Let the potential in Question (5) be $V(x) = \dfrac{1}{2}m\omega^2 x^2$ with ω being an angular frequency. Find $\hat{x}(t)$ using $\hat{x}(0)$ and $\hat{p}(0)$.

(8) Find $[\hat{x}(t), \hat{x}(0)]$ using the result of Question (7).

(END)

Physics B II

【2 pages in total】

II.

In 1924, Einstein received a paper from Bose on photon gases and expanded the research to atomic gases, writing in a letter the same year that 'From a certain temperature on, the molecules "condense" without attractive forces.' Answer the following questions about the statistical mechanics of this phenomenon. Let k_B be the Boltzmann constant, $\beta = 1/(k_B T)$ be the inverse temperature with respect to temperature T, and h be the Planck constant.

(1) Consider a grand canonical ensemble comprising of identical particles with temperature T and chemical potential μ. The grand partition function is expressed as $\Xi = \sum_i e^{-\beta E_i + \beta \mu N_i}$ using the energy E_i of the energy eigenstate i of the whole system and the number of particles N_i. Show that the mean number of particles is expressed as $\langle N \rangle = \dfrac{1}{\beta} \dfrac{\partial}{\partial \mu} \log \Xi$.

(2) Consider an ideal gas of identical particles (ideal Bose gas) that obeys the Bose-Einstein statistics. When the energy of a single-particle energy eigenstate j ($j = 0, 1, 2 \ldots$) is ε_j, the grand partition function is expressed as $\Xi = \prod_{j=0}^{\infty} \Xi_j$ with $\Xi_j = \sum_{n_j=0}^{\infty} e^{-\beta(\varepsilon_j - \mu) n_j}$. Find the mean number of particles occupying the eigenstate j, $\langle n_j \rangle = f(\varepsilon_j, T, \mu)$.

Next, we confine the ideal Bose gas in a three-dimensional box with volume V. Let the mass of a particle be m, the number of particles be N, and the spin be 0. Assuming that the volume V is sufficiently large, single-particle energy eigenstates can be considered to be continuous, and the number of states in the small energy range of $[\varepsilon, \varepsilon + \Delta\varepsilon]$ is given by $D(\varepsilon)\Delta\varepsilon = V \cdot 2\pi(2m/h^2)^{3/2} \sqrt{\varepsilon}\, \Delta\varepsilon$.

(3) Explain why we can consider $\mu \leqq 0$ for the chemical potential.

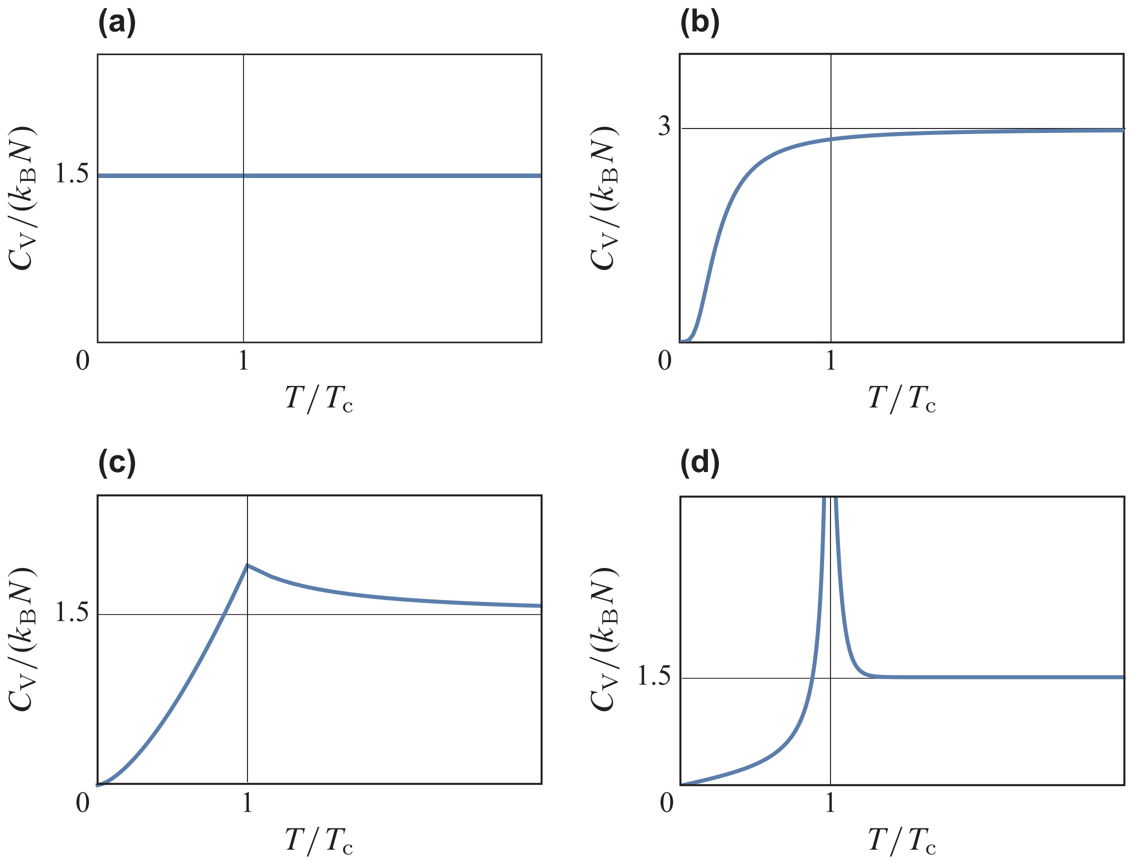
(4) The chemical potential is determined from the condition, $N = \displaystyle\int_0^{\infty} f(\varepsilon, T, \mu) D(\varepsilon)\, \mathrm{d}\varepsilon$. Explain that the chemical potential increases with decreasing temperature.

(5) When the chemical potential is $\mu = 0$, find the temperature. If necessary, you may use $\int_0^{\infty} \sqrt{x}/(e^x - 1)\, \mathrm{d}x = (\sqrt{\pi}/2)\zeta(3/2)$. The zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ takes the value of $2.612\ldots$ for $s = 3/2$; however, you may leave $\zeta(3/2)$ in the result.

(6) When the temperature is lower than the temperature T_c obtained in Question (5), the equation in Question (4) is no longer valid. Let N_0 be the difference between the two sides at $T < T_c$, and express N_0 in terms of N, T, and T_c.

(7) Explain the physical phenomenon that occurs when the temperature is lower than T_c.

(8) Choose the correct graph of the heat capacity at constant volume C_V of this ideal Bose gas from (a) to (d) below. Also, explain the reason why you think it is correct.



(END)