Physics B I

【2 pages in total】

Answer the following questions on quantum mechanics. Let i be the imaginary unit and $\hbar$ be the Planck constant h divided by 2π. For a Hamiltonian $\hat{H}$ the state vector $|\psi(t)\rangle$ evolves in time according to $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$.

A particle of spin 1/2 is in a time-independent uniform magnetic field of strength B_0 applied in the positive direction of the z-axis. In this case, the Hamiltonian due to the interaction between the magnetic moment originating from the spin and the magnetic field can be expressed as

$$\hat{H} = -\omega_0 \hat{S}_z,$$

where $\omega_0 > 0$ is proportional to B_0. When the up- and down-spin states along the z-axis are written as $|\uparrow\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $|\downarrow\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, the spin angular momentum operator is represented with the Pauli matrices as $\hat{S}_i = \frac{\hbar}{2}\sigma_i$ ($i = x, y, z$), where

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

(1) Find the energy eigenvalues of the Hamiltonian $\hat{H}$.

(2) Using $|\uparrow\rangle$ and $|\downarrow\rangle$, express the eigenstate $|S_x;+\rangle$ belonging to the eigenvalue $+\hbar/2$ of the spin angular momentum operator in the x-axis direction $\hat{S}_x$.

When the directions of the spin and the magnetic field are not parallel, the direction of the spin undergoes a precession. Suppose that the spin angular momentum is $S_x = +\hbar/2$ at time $t = 0$.

(3) Using $|\uparrow\rangle$ and $|\downarrow\rangle$, express the state vector of the spin $|\psi(t)\rangle$ at time $t \geqq 0$.

(4) By considering $\langle S_x;+|\psi(t)\rangle$, find the probability of observeing a state with spin angular momentum of $S_x = +\hbar/2$ at time $t \geqq 0$.

(5) Find the expectation value of the spin angular momentum in the x-axis direction $\langle \hat{S}_x \rangle$ as a function of time t.

When an oscillating magnetic field sufficiently weak compared to B_0, $B_1 \cos\omega t$, is applied in the x-axis, the Hamiltonian can be expressed in the form,

$$\hat{H} = -\omega_0 \hat{S}_z - 4\lambda\cos(\omega t)\hat{S}_x, \quad |\lambda| \ll \omega_0.$$

Suppose the spin angular momentum at time $t = 0$ is $S_z = -\hbar/2$.

(6) The spin state at time $t \geqq 0$ is written as $|\psi(t)\rangle = C_\uparrow(t)e^{+i\omega_0 t/2}|\uparrow\rangle + C_\downarrow(t)e^{-i\omega_0 t/2}|\downarrow\rangle$. Find the differential equations satisfied by $C_\uparrow(t)$ and $C_\downarrow(t)$. For simplicity of computation, you may neglect the term including the high frequency components, $e^{\pm i(\omega+\omega_0)t}$.

(7) Expand the coefficients in Question (6) as $C_{\uparrow\downarrow}(t) = C^{(0)}_{\uparrow\downarrow}(t) + \lambda C^{(1)}_{\uparrow\downarrow}(t) + \cdots$ with respect to λ. By comparing the powers of λ appearing on both sides of the differential equations, find the differential equation satisfied by $C^{(0)}_\uparrow(t)$ and $C^{(1)}_\uparrow(t)$.

(8) By using the first-order perturbative approximation, find the probability of observing a state with spin angular momentum $S_z = +\hbar/2$ at time $t \geqq 0$.

(END)

Physics B II

【1 page in total】

Answer the following questions on statistical mechanics. Let k_{B} be the Boltzmann constant, $\beta = 1/(k_{\mathrm{B}}T)$ be the inverse temperature with respect to temperature T, and $\hbar$ be the Planck constant h divided by 2π.

Consider the model proposed by Einstein to explain that the specific heat of a solid is zero at temperature $T = 0$. In this model, the lattice vibrations in a crystal consisting of N atoms are considered as a set of $3N$ independent one-dimensional harmonic oscillators with angular frequency, ω.

Suppose this system is in thermal equilibrium at temperature T. Because the energy eigenvalues of a harmonic oscillator are expressed as $\varepsilon_n = \hbar\omega(n + 1/2)$ with $n = 0, 1, 2, \ldots$, the partition function of the whole system can be expressed as $Z = z^{3N}$ with $z = \sum_{n=0}^{\infty} e^{-\beta\varepsilon_n}$.

(1) Show that the mean energy of the system in thermal equilibrium is expressed as $\langle E \rangle = -\dfrac{\partial}{\partial\beta} \log Z$.

(2) Find the mean energy of the system, $\langle E \rangle$.

(3) Find the heat capacity at constant volume, C_{V}.

(4) In the high-temperature limit of $k_{\mathrm{B}}T \gg \hbar\omega$, show that the result of Question (3) is the heat capacity obtained using classical mechanics. Also explain its physical meaning.

(5) In the low-temperature limit of $k_{\mathrm{B}}T \ll \hbar\omega$, show that the result of Question (3) approaches zero as the temperature decreases. Also explain the physical reasons for the difference from the results obtained using classical mechanics.

(6) This Einstein model cannot explain the experimental result that the specific heat in the low-temperature limit is proportional to the cube of the temperature T. Suggest how the model can be improved.

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